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Existence of immersed tori in manifolds of nonpositive curvature

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1. Introduction

In this paper we study complete Riemannian manifolds of nonpositive sectional curvature $K \leqq 0$. The universal covering X of such a manifold M is diffeomorphic to euclidean space and M can be identified with X/Γ, where $\Gamma \cong \pi_1(M)$ is the group of decktransformations. The flat torus theorem [GW], [LY] states: *If M is compact, then M contains a totally geodesic immersed k-dimensional flat torus if and only if Γ contains a subgroup isomorphic to $\mathbb{Z}^k$.*

The lift of a flat k-torus to X is a k-flat in X, i.e. a complete totally geodesic and isometric embedding $F : \mathbb{R}^k \to X$. We are interested in the opposite question: *Does the existence of a k-flat in X imply the existence of a flat k-torus in M?*

The main result of this paper is an affirmative answer in the case that the flat has codimension $\leqq 2$ and the metric is real analytic.

Theorem 1. *Let M be a complete real analytic n-dimensional Riemannian manifold with sectional curvature $-b^2 \leqq K \leqq 0$ and of finite volume. Let $\pi : X \to M$ be the universal covering space. If X contains a k-flat F, $k \geqq n-2$ such that $\pi(F)$ is contained in a compact subset of M, then M contains a totally geodesic flat immersed k-torus.*

The analyticity condition is crucial. If we assume M only to be smooth (i.e. C^∞-smooth) and complete of finite volume, then there are counterexamples:

Theorem 2. *For every integer $n \geqq 4$, there exists a complete smooth n-dimensional manifold M with sectional curvature $-b^2 \leqq K \leqq 0$ and finite volume, such that the universal covering X of M contains an $(n-2)$-flat F but there are no immersed flat tori in M. The flat F satisfies that $\pi(F)$ is contained in a compact subset of M, where $\pi : X \to M$ is the canonical projection.*

On the other hand we know no compact counterexamples. In the codimension 1 situation we prove the stronger result that every $(n-1)$-flat F in X covers an immersed torus or we are in an essentially 2-dimensional situation.

Theorem 3. *Let $M = X/\Gamma$ be a real analytic n-dimensional complete manifold with curvature $-b^2 \leqq K \leqq 0$ and finite volume. Then every $(n-1)$-flat in X covers a flat torus or X splits isometrically as $X' \times \mathbb{R}^{n-2}$ with a 2-dimensional factor X'.*

In the codimension 1 situation one can prove the existence of a flat torus also in the C^∞-category. The following theorem holds: *Let $M = X/\Gamma$ be a smooth n-dimensional complete Riemannian manifold of curvature $-b^2 \leqq K \leqq 0$ and finite volume with finitely generated fundamental group. If X contains an $(n-1)$-flat, then there exists an immersed flat $(n-1)$-torus in M.* Since the proof of this result is very technical we only refer to [S].

The results of our paper can be viewed in the larger context of the following question:

To what extend is the Tits-geometry of $X(\infty)$ determined by the fundamental group of a compact (or finite volume) quotient M of X?

For the notion and basic results on the Tits-geometry see [BGS], § 4. Roughly speaking, the Tits-geometry describes the asymptotical flatness of X. As a special case the Tits-geometry is degenerated if and only if M is a visibility manifold (see section 2 for a definition). By a result of Eberlein [E2] a compact manifold M of curvature $K \leqq 0$ satisfies the visibility axiom if and only if the covering X contains a 2-flat. Thus Theorem 1 has the following immediate

Corollary 1. *Let M be a compact real analytic manifold with $K \leqq 0$ and dimension $\leqq 4$. Then M is a visibility manifold if and only if every abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.*

Our argument can be modified to obtain some results also for higher codimension.

Corollary 2. *Let $M = X/\Gamma$ be compact, real analytic with $K \leqq 0$. Let $n = \dim M \geqq 3$. If X contains a k-flat with $k \geqq n/2$, then in M there exists an immersed 2-torus. In particular Γ contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

The visibility property of a compact manifold is a homotopy invariant by a result of Eberlein [E1]. More generally the existence of a k-flat in the universal covering X is a homotopy invariant for compact manifolds by [AS]: *Let M and M^* be compact manifolds with $K \leqq 0$ and isomorphic fundamental groups. Let X and X^* be the universal coverings. Then X contains a k-flat if and only if X^* contains a k-flat.*

The homotopy invariance of the maximal dimension of an immersed torus follows from the flat torus theorem. Our result implies that both invariants coincide in the codimension $\leqq 2$ case at least if the metric is real analytic.

We give a brief indication of the main step in the proof of Theorem 1. Let $F \subset M = X/\Gamma$ be a flat of codimension 2. If F does not cover a flat torus then the set γF, $\gamma \in \Gamma$ is not discrete and there exists a sequence F_i of flats in X converging to a limit flat F_∞. We will prove in Proposition 1 that the existence of such a sequence in X (and the existence of a compact quotient) implies the existence of a flat F^* and a hyperplane $H^* \subset F^*$ such that H^* has a parallel outside of F^*. Thus the set P_{H^*} of all parallels to H^* is an $(n-1)$-dimensional convex subset of X isometric to $Q \times \mathbb{R}^{n-3}$. Note that by analyticity P_{H^*} is a complete hypersurface. Now one can use special hypersurface arguments similar to the proof of Theorem 3 to finish the argument. The proposition itself does not use the codimension 2 condition and is true in larger generality. It may be useful also in other situations.

The organisation of the paper is as follows. After the preliminary section 2 we describe the example of Theorem 2. In section 4 we study the codimension 1 situation and prove Theorem 3. In section 5 we prove Proposition 1 which is essential for the main result which is contained in section 6. In a final section we discuss some examples, prove the corollaries, and conclude the paper with some general remarks.

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2. Preliminaries

We refer in general to [EO], [BGS], [BE]. Let M be a complete Riemannian manifold of nonpositive curvature with universal covering $\pi: X \to M$. The distance function on X is denoted by d. This distance function is convex. On the ideal boundary $X(\infty)$ we consider the cone topology [EO]. For a geodesic $c: \mathbb{R} \to X$ let $c(\infty)$ and $c(-\infty)$ be its endpoints at infinity. More generally, for any complete totally geodesic submanifold H of X we consider the boundary $H(\infty) \subset X(\infty)$. A complete totally geodesic submanifold H_0 is called parallel to H, if $H(\infty) = H_0(\infty)$. This is equivalent to the fact that the Hausdorff distance Hd between H and H_0 is bounded, i.e. $\mathrm{Hd}(H, H_0) < \infty$. Two parallel totally geodesic submanifolds bound a convex region isometric to $H \times [0, a]$, where $a = \mathrm{Hd}(H, H_0)$ (Sandwich-lemma). In general the set P_H of all points in X which lie on parallels to H form a convex subset which splits isometrically as $P_H = H \times Q$ where Q is a convex subset of X. If X is real analytic the set P_H is complete.

The manifold X satisfies the visibility property, if for any two different points $z \neq w$ in $X(\infty)$ there exists a geodesic $c: \mathbb{R} \to X$ with $c(\infty) = z$ and $c(-\infty) = w$.

We call X irreducible, if X cannot be written as a Riemannian product $X_1 \times X_2$ with factors of positive dimension.

In this section we reduce the statements of Theorem 1 and Theorem 3 to the case that X is irreducible. We first handle the case of an euclidean de Rham factor. Therefore we need the following result ([E3], corollary 2, compare also the discussion in [BE], section 1): *Let $M = X/\Gamma$ be of finite volume with finitely generated fundamental group Γ. Assume that the euclidean de Rham factor of X has dimension s, i.e. $X = \mathbb{R}^s \times X_1$, X_1 without euclidean factor. Then there exists a subgroup Γ^* of finite index in Γ which is of the form $\mathbb{Z}^s \times \Gamma_1^*$ where $\mathbb{Z}^s$ operates trivially on X_1 and as a lattice on $\mathbb{R}^s$. Every $\gamma \in \Gamma_1^*$ operates as a translation on $\mathbb{R}^s$ and without fixed points on X_1. The quotient X_1/Γ^* has finite volume.*

We can apply this result for analytic manifolds with curvature $-b^2 \leqq K \leqq 0$ and finite volume, since such a manifold has finitely generated fundamental group by a result of Gromov [BGS], Lecture III. If F is a flat of maximal dimension in X, then F splits as $\mathbb{R}^s \times F_1$ in $X = \mathbb{R}^s \times X_1$ where F_1 is a flat of maximal dimension in X_1.

If Theorem 1 holds for manifolds without euclidean factor, then there exists a flat F_1' with $\dim F_1' = \dim F_1$ and a subgroup Δ_1^* of Γ_1^* which leaves F_1' invariant and operates with compact quotient on this flat. Then the subgroup $\mathbb{Z}^s \times \Delta_1^* \subset \Gamma$ operates with compact quotient on $\mathbb{R}^s \times F_1'$.

In a similar way one can reduce Theorem 3 to the case of manifolds without euclidean factor.

If $M = X/\Gamma$ has finite volume and splits isometrically as $X_1 \times X_2$ and has no euclidean factor, then the following alternative holds ([E4], compare also the discussion in section 1 of [BE]): *Either X is a symmetric space or there exists a finite index subgroup Γ^* that is a direct product $\Gamma_1^* \times \Gamma_2^*$ and X/Γ^* is a Riemannian product $X_1/\Gamma_1^* \times X_2/\Gamma_2^*$ that finitely covers X/Γ.*

Note that Theorem 1 is true for symmetric spaces (in the case of symmetric spaces it is even known that the flats which cover tori are dense in the set of all flats of maximal dimension [M]). If, in the nonsymmetric case F is a codimension 2 flat in X, then $F = F_1 \times F_2$ where the F_i are codimension 1 flats in X_i (since X has no euclidean factor). Thus if M is finitely covered by $X_1/\Gamma_1^* \times X_2/\Gamma_2^*$ then by induction on the factors there exists a codimension 2 torus in M. In a similar way one can reduce Theorem 3 to the case that X is irreducible.

In this paper we study flats in X and want to prove the existence of closed tori in M. Therefore we consider more generally a complete totally geodesic submanifold $H \subset X$ where X covers a manifold $M = X/\Gamma$ of finite volume. Let $\Gamma_H := \{\gamma \in \Gamma \mid \gamma H = H\}$.

Lemma 1. *Either H/Γ_H has finite volume or for all $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that $\gamma H \neq H$ and $d(H, \gamma H) < \varepsilon$.*

Proof. Assume that the statement does not hold for a fixed $\varepsilon > 0$. Then for the $\varepsilon/2$-tube $D = T_{\varepsilon/2} H$ it follows:

$$\gamma D \cap D \neq \emptyset \Rightarrow \gamma \in \Gamma_H .$$

Thus D/Γ_H is injectively imbedded in M and therefore has finite volume. It follows that H/Γ_H has finite volume. □

In the case that H is a flat it is well known (from the theory of crystallographic groups) that the finiteness of the volume of H/Γ_H is equivalent to the compactness.

Sometimes we consider the set of tangent spaces of a totally geodesic submanifold. Let therefore $G_k(X)$ be the bundle of k-planes, i.e. the fiber of $G_k(X)$ in $x \in X$ is the set of all k-planes in $T_x X$. On $G_k(X)$ we have an induced distance function d^k. The tangent bundle TH of a k-dimensional complete totally geodesic submanifold H of X is a submanifold of $G_k(X)$. One can easily generalize Lemma 1 to

Lemma 2. *Either H/Γ_H has finite volume or for all $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that $\gamma H \neq H$ and $d^k(TH, \gamma_* TH) < \varepsilon$. Here γ_* is the differential of γ.*

3. The example of Theorem 2

The example is a modification of an example of Heintze (compare also section 7). We start with an n-dimensional noncompact hyperbolic ($K \equiv -1$) manifold of finite volume. A cusp of M has metrically the structure of a warped product $(0, \infty) \times_g \mathbb{R}^{n-1}/\Gamma$ with the warping function $g: (0, \infty) \to \mathbb{R}$, $g(t) = e^{-t}$, i.e. the metric is given by $dt^2 + g^2(t)\, ds^2$, where ds^2 is the standard metric on $\mathbb{R}^{n-1}$. The group Γ operates with compact quotient on $\mathbb{R}^{n-1}$. It is known that there are examples such that Γ is a group of translations.

We write $(0, \infty) \times_g \mathbb{R}^{n-1}$ in a formal way as $Y \times_f \mathbb{R}$, where $Y = (0, \infty) \times_g \mathbb{R}^{n-2}$ and $f: Y \to \mathbb{R}$ is defined by $f(t, p) = e^{-t}$. Now we replace this metric by

$$((0, \infty) \times_{\bar{g}} \mathbb{R}^{n-2}) \times_f \mathbb{R}$$

where $\bar{g}: (0, \infty) \to \mathbb{R}$ is a new convex warping function which coincides with g for small t and is a positive constant for $t \geqq t_0$. Note that the translations of $\mathbb{R}^{n-1}$ are isometries with respect to the new metric. Hence we obtain a new metric on M and a new Riemannian manifold $\bar{M}$.

We show that the curvature is nonpositive: By the curvature formula for warped products [BON], 7.7, we have to show that $\bar{g}$ is convex on $(0, \infty)$ and that f is convex on $(0, \infty) \times_{\bar{g}} \mathbb{R}^{n-2}$. By construction $\bar{g}$ is convex. To prove that f is convex, note that $(0, \infty) \times_{\bar{g}} \mathbb{R}^{n-2}$ is a subset of the complete manifold $\bar{Y} = \mathbb{R} \times_{\bar{g}} \mathbb{R}^{n-2}$ where $\bar{g}(s) = g(s)$ for $s < 0$. $\bar{Y}$ is a Hadamard manifold. One checks easily that the curves $c_p: \mathbb{R} \to \bar{Y}$ with $c_p(t) = (t, p)$ are asymptotic unit speed geodesics with common Busemann function $b(t, p) = -t$. Busemann functions are convex. Let $c: [0, 1] \to \bar{Y}$ be a geodesic. Since $s \mapsto b \circ c(s)$ is convex, also $s \mapsto e^{b \circ c(s)} = f \circ c(s)$ is convex. Therefore f is convex.

It is also easy to check that the new metric has a lower curvature bound. Furthermore, since the last direction shrinks exponentially, it follows that $\mathrm{vol}(\bar{M}) < \infty$. Note that the sets $(\{t\} \times \mathbb{R}^{n-2}) \times \{p\}$ are $(n-2)$-flats for $t \geqq t_0$. The projection of these flats stays in a compact subset of $\bar{M}$.

In this construction we have the freedom to choose $\mathbb{R}^{n-2}$ as an arbitrary affine subspace of $\mathbb{R}^{n-1}$. Thus we can choose $\mathbb{R}^{n-2}$ to be irrational for the lattice Γ. Then the flats do not close up in $\bar{M}$. It is also not difficult to check that there are no other immersed tori in $\bar{M}$.

4. Codimension 1 flats in analytic manifolds

We will prove Theorem 3. By the discussion in section 2 we can assume that X is irreducible. Thus it remains to prove: *Let M be of dimension $n \geqq 3$, analytic, $-b^2 \leqq K \leqq 0$ and without euclidean factor. Then every $(n-1)$-flat in X covers a flat torus in M.*

Lemma 3. *Different $(n-1)$-flats in X cannot intersect.*

Proof. Let F_1, F_2 be $(n-1)$-flats with $x \in F_1 \cap F_2$. If $F_1 \neq F_2$, then $W = F_1 \cap F_2$ is a hyperplane in F_1 and in F_2. Both flats are fibered by parallels to W. Thus the set P_W of all parallels is n-dimensional and by analyticity $X = P_W = X' \times \mathbb{R}^{n-2}$. This contradicts the condition that X has no euclidean factor. □

Let $B := \bigcup \{F \mid F \subset X \text{ is an } (n-1)\text{-flat}\}$. Then B is a closed subset of X. We consider separately the cases:

a) $B = X$ and b) $B \neq X$.

In case a) through every point $x \in X$ there exists one and because of Lemma 3, exactly one $(n-1)$-flat. It follows that we have a Γ-invariant foliation of X by codimension 1 flats. Since X has no euclidean de Rham factor this contradicts to the following lemma.

Lemma 4. *Let $M = X/\Gamma$ be a smooth complete manifold with $K \leqq 0$ and finite volume. If there exists a Γ-invariant foliation of X by complete totally geodesic hypersurfaces, then X splits isometrically as $X' \times \mathbb{R}$ and the leaves of the foliation are the hypersurfaces $X' \times \{t\}$, $t \in \mathbb{R}$.*

Proof. We have to show, that any two leaves of the foliation have finite Hausdorff distance. Then the result follows from the Sandwich lemma. Therefore let us assume to the contrary that there are two leaves F_0 and F_1 in the foliation with $\mathrm{Hd}(F_0, F_1) = \infty$. Let Z be the convex region bounded by F_0 and F_1 and choose a geodesic $c: [0, 1] \to Z$ with $c(0) \in F_0$ and $c(1) \in F_1$. Since $\dot{c}(0)$ is transversal to F_0 and the foliation is totally geodesic, c meets every leaf at most once. Let F_t be the leaf with $c(t) \in F_t$ for $t \in [0, 1]$. One checks easily that $Z = \bigcup \{F_t | t \in [0, 1]\}$. Since $\mathrm{Hd}(F_0, F_1) = \infty$ there exists a geodesic ray $h: [0, \infty) \to Z$ with $h(0) \in F_0$ and $d(h(t), F_0) \to \infty$, $d(h(t), F_1) \to \infty$.

It is well known that the initial vectors of closed geodesics are dense in the unit tangent bundle of M. Thus there exists a geodesic $g: \mathbb{R} \to X$ which is the axis of an isometry $\gamma \in \Gamma$ (i.e. $\gamma g(t) = g(t + \omega)$ for an $\omega > 0$) with

$$g(0) \in F_0,\ g([0, \infty)) \subset Z,\ d(g(t), F_0) \to \infty \quad \text{and} \quad d(g(t), F_1) \to \infty.$$

Define $r(t)$ such that $g(t) \in F_{r(t)}$ for $t \in [0, \infty)$. Then $r(t)$ is strictly increasing with $r(0) = 0$ and $0 < \tau := \lim\limits_{t \to \infty} r(t) \leqq 1$. Note that $\gamma^n F_0 \to F_\tau$ uniformly on compact sets and F_τ is invariant under γ. Thus $d(g(t), F_\tau) = d(g(t + \omega), F_\tau)$ for all $t \in \mathbb{R}$ and it follows that the function $t \mapsto d(g(t), F_\tau)$ is constant as a bounded convex function. Since $g(0) \in F_0$ and $\dot{g}(0)$ transversal to F_0, it follows that $F_\tau \cap F_0 \neq \emptyset$. This is a contradiction. □

We also need the following result:

Lemma 5. *There exists $\varepsilon = \varepsilon(b) > 0$ with the property: Let X be a Hadamard manifold with curvature $-b^2 \leqq K \leqq 0$. Let F and F' be complete totally geodesic hypersurfaces in X with $F \cap F' = \emptyset$, $x \in F$ and $d(x, F') < \varepsilon$. Let $c: [0, a] \to X$ be the minimal geodesic from x to F' and $N(x)$ the normal vector to F in x with $\sphericalangle(N(x), \dot{c}(0)) < \pi/2$. Then $\sphericalangle(N(x), \dot{c}(0)) \leqq \pi/4$ and the ray $h: [0, \infty) \to X$ with $\dot{h}(0) = N(x)$ meets F'.*

Proof. In the hyperbolic plane of constant curvature $-b^2$ we consider the ideal triangle with angles $\pi/2$ and $\pi/4$. Let $\varepsilon(b)$ be the length of the finite edge of this triangle. Let F, F', x and c as in the assumption. If $\sphericalangle(N(x), \dot{c}(0)) > \pi/4$, then there is a ray $g: [0, \infty) \to F$ with $g(0) = x$ and $\sphericalangle(\dot{g}(0), \dot{c}(0)) \leqq \pi/4$. We consider the ideal triangle x, $c(a)$, $g(\infty)$. Since g does not meet F' we have $\sphericalangle_{c(a)}(x, g(\infty)) \leqq \pi/2$. A comparison with the space of constant curvature implies, that c has length $\leqq \varepsilon$.

If the ray h does not intersect F', then consider the ideal triangle x, $c(a)$, $h(\infty)$. With the comparison above we obtain a contradiction. □

Remark. The lemma says that the manifold F' can be described in a canonical way as a graph over F in the set, where it is near to F.

We consider now the case b) and assume that $B \neq X$. Let W be a connected component of $X \setminus B$. Then W is an open convex subset of X and the boundary consists of (finitely or infinitely many) flats. Let F be a boundary flat of W.

Lemma 6. *There exists a subgroup A of Γ isomorphic to $\mathbb{Z}^{n-2}$, which leaves F invariant and operates as a group of translations on F.*

Proof. Let $Q := \{x \in W \mid d(x, F) < d(x, F')$ for all other boundary flats F' of $W\}$. Since B is Γ-invariant, W and Q are precisely invariant under Γ, i.e. $\gamma Q \cap Q \neq \emptyset \Rightarrow \gamma Q = Q$. Let $\Gamma_Q := \{\gamma \in \Gamma \mid \gamma Q = Q\}$. By construction Γ_Q leaves F invariant and operates as a group of isometries of $F = \mathbb{R}^{n-1}$. We have to show that $k := \operatorname{rank} \Gamma_Q$ is at least $(n-2)$.

From the structure theory of flat manifolds it is well known that there exists an affine subspace H of dimension k in F, which is invariant under Γ_Q. Furthermore there exists $\zeta > 0$ such that $d(p, \gamma p) \geqq 2\zeta$ for all $\gamma \in \Gamma_Q \setminus \mathrm{id}$ and $p \in F$.

Let $T_R(H) = \{x \in F \mid d(x, H) \leqq R\}$, and let N be the unit normal field of F pointing towards W. For $x \in F$ let $\phi(x) = \inf\{d(x, F') \mid F' \neq F$ is a boundary flat of $W\}$.

We claim that for every $\varepsilon > 0$ there exists an $R > 0$ such that $\phi(x) < \varepsilon$ for all $x \in F \setminus T_R(H)$.

If we assume the contrary then there exists a sequence $p_i \in F$ with $d(p_i, H) \longrightarrow \infty$, such that $\phi(p_i) \geqq \varepsilon$. We can assume that

$$d(p_{i+1}, H) > d(p_i, H) + 1. \quad \text{Let} \quad \eta = \min(\varepsilon/4, 1/4, \zeta).$$

Then the η-balls at the points $\exp(\eta \cdot N(p_i))$ are contained in Q. We denote this ball with B_i. If $\gamma B_i \cap B_j \neq \emptyset$ for $\gamma \in \Gamma$ then $\gamma \in \Gamma_Q$ and $d(\gamma p_i, p_j) < 2\eta$. Since γ leaves the distance to H invariant, we have $p_i = p_j$ and $\gamma = \mathrm{id}$ by the choice of ζ. It follows $\operatorname{vol}(M) \geqq \sum_{i=1}^{\infty} \operatorname{vol}(B_i) = \infty$, which is a contradiction.

In particular there exists $R > 0$ such that $\phi(x) < \varepsilon(b)$ on $F \setminus T_R(H)$ where $\varepsilon(b)$ is the constant of Lemma 5. Let us assume $k < n-2$. Then $F \setminus T_R(H)$ is connected. By Lemma 5 the geodesic $\exp_x t \cdot N(x)$ intersects an other boundary component F' of W for every $x \in F \setminus T_R(H)$. By convexity this geodesic intersects only one other boundary component. Thus F' cannot vary, if we vary x. By connectedness this boundary component is F' for all $x \in F \setminus T_R(H)$. In particular $d(x, F') < \varepsilon$ for all $x \in F \setminus T_R(H)$. Since $T_R(H)$ is invariant under Γ_Q, also F' is Γ_Q invariant. Since $T_R(H)/\Gamma_Q$ is compact, it follows that $d(\ , F')$ is bounded on F. It follows that F and F' are parallel. By analyticity $X = F \times \mathbb{R}$, i.e. X is isometric to $\mathbb{R}^n$ which is a contradiction to our assumption.

We are now able to complete the proof:

Proof of Theorem 3. Let $F \subset X$ be an $(n-1)$-flat. We claim that F is isolated, i.e. there does not exist a sequence $F_i \neq F$ of flats with $F_i \longrightarrow F$.

Before we prove this claim we show that it implies Theorem 3. Since F is isolated, F is the boundary flat of a connected component of $X \setminus B$ where B is as above the set of all flats. Let Γ_F be the subgroup of Γ which leaves F invariant. We have to prove that Γ_F operates with compact quotient on F, i.e. $\operatorname{rank} \Gamma_F = n-1$. By Lemma 6 we know already that $\operatorname{rank} \Gamma_F \geqq n-2$. If $\operatorname{rank} \Gamma_F = n-2$ then there exists an affine hyperplane H^{n-2} in F^{n-1} invariant under Γ_F. The hyperplane H devides F in two components F^+

and F^-. By considering subgroups of finite index we may assume without loss of generality that every $\gamma \in \Gamma_F$ leaves the two components of $X \backslash F$ and the two components of $F \backslash H$ invariant.

It follows in particular that Γ_F leaves W invariant. As in the proof of Lemma 6 there exists $R > 0$ such that $\phi(x) < \varepsilon(b)$ on $F \backslash T_R(H)$, where ϕ and $\varepsilon(b)$ are defined as above. Now $F^+ \backslash T_R(H)$ is connected and it follows that there is a unique other boundary component F' of W such that $d(x, F') < \varepsilon$ for all $x \in F^+ \backslash T_R(H)$.

Therefore Γ_F leaves also F' invariant and there exists a hyperplane H' in F' invariant under Γ_F. Note that H' is parallel to H. Thus P_H has dimension n and it follows that $X = P_H = X' \times \mathbb{R}^{n-2}$, which proves the theorem.

It remains to prove the claim: Let us assume that there is a sequence $F_i \longrightarrow F$. In the Grassmann bundle $G_{n-1}(X)$ we consider the subset Y of all tangent planes to flats. For $\sigma \in G_{n-1}(X)$ let S_σ be the unit sphere of σ. If σ is tangent to an $(n-1)$-flat, then the volume of $\exp(S_\sigma)$ equals ω_{n-2}, where ω_{n-2} ist the volume of the unit $(n-2)$-sphere. If $\operatorname{vol}(\exp(S_\sigma)) = \omega_{n-2}$, then it follows from [BGS], § 1.E, that the unit ball in σ is mapped totally geodesically onto a flat $(n-1)$-ball in X. By analyticity σ is tangent to an $(n-1)$-flat. Thus $Y = f^{-1}(\omega_{n-2})$ where f is the analytic function $f(\sigma) = \operatorname{vol}(\exp(S_\sigma))$, i.e. Y is an analytic subset of $G_{n-1}(X)$. Let $p: G_{n-1} \longrightarrow X$ be the canonical projection. Lemma 3 implies that $p_{|Y}: Y \longrightarrow X$ is injective. Thus the dimension of Y is $\leqq n$. The set Y is a manifold almost everywhere. Let $\sigma \in Y$ be a point, such that Y is a manifold near σ. Note that σ is tangent to a flat F and thus contained in the $(n-1)$-dimensional manifold TF consisting of all tangent planes to F. If the dimension of Y near σ equals $(n-1)$, then there exists an open neighborhood U of σ with $U \cap Y = U \cap TF$. Then there cannot exist a sequence $F_i \longrightarrow F$, $F_i \neq F$ and thus TF is isolated in Y. Since analytic sets can be stratified (comp. e.g. [H]), in every connected component of Y there are points where Y is locally a manifold. If the local dimension is $n-1$ everywhere, the above argument shows that Y is a discrete union of components of the form TF.

Since there exists an accumulation flat by assumption, Y has somewhere the dimension n. It follows that there exists an open set U in $G_{n-2}(X)$ such that $U \cap Y$ is an n-ball. Since $p_{|Y}: Y \longrightarrow X$ is injective, $p(U \cap Y)$ covers an open subset D of X such that through every point $x \in D$ there exists exactly one $(n-1)$-flat. The uniqueness implies that the flats depend continuously from x.

We choose $x \in D$ and denote the flat through x by F_0. Let $c: [0, 1] \longrightarrow D$ be a geodesic with $c(0) = x$ and $\dot{c}(0)$ is transversal to F_0. Then c can meet any flat only once. Let F_t be the flat with $c(t) \in F_t$. It follows easily that the set $Z = \bigcup \{F_t \mid t \in [0, 1]\}$ is a convex region bounded by F_0 and F_1. The same argument as in the proof of Lemma 4 shows that either $\operatorname{Hd}(F_0, F_1) < \infty$ or there exists F_τ which intersects F_0. In the first case X splits because of the analyticity as $F_0 \times \mathbb{R}$, in the second case we have a euclidean factor by Lemma 3. This proves the claim. □

5. Accumulation of flats

Proposition 1. *Let X be a smooth Hadamard manifold, and let F_i be a sequence of different k-flats in X which converge uniformly on compact subsets to a k-flat F. If the isometry group of X acts uniformly on F, then there exists also a k-flat F^* which contains* *a hyperplane $H^* \subset F^*$ such that $\dim P_{H^*} > k$. In particular, if X is real analytic, then P_{H^*} is isometric to $\mathbb{R}^{k-1} \times Q$, where Q is complete and of dimension $\geqq 2$.*

Remark. The condition that the isometry group acts uniformly on F says that there exists a fixed compact subset $D \subset X$ such that for given $x \in F$ there exists an isometry γ of X with $\gamma x \in D$.

Proof. On F we consider the distance functions $f_i = d(\ , F_i)_{|F}$. If f_i is constant on F, then F_i is parallel to F and the set of parallels P_F has dimension $\geqq (k+1)$. The conclusion of the proposition is clear in this case.

Thus we assume without loss of generality that f_i is not constant. We fix an origin $p \in F$. Since $F_i \to F$ we see $f_i(p) \to 0$ and we can assume that $f_i(p) \leqq 1$. Let A_i be the radius of the largest distance ball $B_i = B_{A_i}(p)$ of p in F, such that $B_i \subset \{f_i \leqq 1\}$. Since $F_i \to F$ we see $A_i \to \infty$. On the other hand $A_i \neq \infty$ since f_i is not bounded. Let q_i be a point in ∂B_i with $f_i(q_i) = 1$. Let H_i be the hyperplane in F tangent to ∂B_i in q_i. By construction H_i is also tangent to $\{f_i = 1\}$ and thus $f_i \geqq 1$ on H_i by the convexity of f_i. Let D_i be the ball in H_i at the point q_i with radius $r_i := \sqrt{(A_i+1)^2 - A_i^2}$. By Pythagoras' theorem $d(x, \partial B_i) \leqq 1$ for all $x \in D_i$ and hence $f_i(x) \leqq 2$ by the triangle inequality. (Note that we have used in an essential way that F is euclidean.) Since $A_i \to \infty$ also $r_i \to \infty$. Since the isometry group operates uniformly on F, there are isometries γ_i such that $\gamma_i(q_i)$ are contained in a compact fundamental domain. By considering subsequences we can assume that $\gamma_i D_i \to D^*$ (where D^* is a $(k-1)$-flat in X), and since $1 \leqq f_i(x) \leqq 2$ on D_i, $\gamma_i F_i$ converges to a k-flat F^* such that D^* has bounded, hence by convexity constant distance to F^*. Thus in F^* there exists a parallel H^* to D^*.

6. Proof of Theorem 1

By the discussion of section 2 we can assume that X^n is irreducible, i.e. X has no euclidean factor and is not a product.

Let $F \subset X$ be a flat of codimension 2 such that $\pi(F) \subset M$ is contained in a compact subset of M. If $\pi(F)$ is not an immersed torus, then the set γF, $\gamma \in \Gamma$ is not discrete and there exists an accumulation flat in X. By Proposition 1 there exists a complete totally geodesic submanifold $W \subset X$ isometric to $\mathbb{R}^{n-3} \times Q$ with $\dim Q \geqq 2$. Since X has no euclidean factor, $\dim Q = 2$. We call a complete totally geodesic submanifold W in X *singular*, if W has dimension $(n-1)$ and splits isometrically as $\mathbb{R}^{n-3} \times Q$. We can assume that X contains no $(n-1)$-flats, thus Q is not isometric to $\mathbb{R}^2$.

We first study the possible intersection of different singular submanifolds. Let $W' \neq W$ be singular submanifolds with $S = W' \cap W \neq \emptyset$. Then S is a totally geodesic hyperplane in W. Note that W is isometric to $Q \times \mathbb{R}^p$, with $\dim Q = 2$ and Q not isometric to $\mathbb{R}^2$. We claim that every complete totally geodesic hyperplane in $Q \times \mathbb{R}^p$ respects the splitting, i.e. either $S = \mathbb{R} \times \mathbb{R}^p$ where $\mathbb{R}$ is a geodesic in Q, or $S = Q \times \mathbb{R}^{p-1}$ where $\mathbb{R}^{p-1}$ is an affine subspace of $\mathbb{R}^p$.

To prove this claim, let N be a normal unit vectorfield to S and let

$$p_1 : Q \times \mathbb{R}^p \to Q$$

be the canonical projection. Note that N is a global parallel field on S which implies that $p_{1*}N$ is a global parallel vectorfield on the convex set $p_1(S) \subset Q$. Since Q is not isometric on $\mathbb{R}^2$, there are no global parallel vectorfields on Q. Thus either $p_{1*}N \equiv 0$ (and then $S = Q \times \mathbb{R}^{p-1}$) or $p_1(S)$ is a geodesic in Q (and $S = \mathbb{R} \times \mathbb{R}^p$). In the first case S is not flat, in the second case S is flat. Thus the type of splitting is the same in W and W'.

If $S = Q \times \mathbb{R}^{p-1}$ then W and W' are foliated by parallels to Q. This implies that $\dim P_Q = n$ and X splits as $Q \times X'$. This is ruled out by our assumption. Thus

$$S = \mathbb{R} \times \mathbb{R}^p.$$

This implies that the euclidean factor of W at x is completely contained in W' and vice versa.

If $\dim M \geqq 5$, then this implies that the euclidean factors of W and W' have a geodesic c in common. Then $W \subset P_c$ and $W' \subset P_c$ which implies $\dim P_c = n$ and $X = P_c = \mathbb{R} \times X'$ in contrast to our assumptions. This argument shows that for $\dim X \geqq 5$ different singular submanifolds cannot intersect. Let $\dim X = 4$ and $W \neq W'$ be singular submanifolds with $S = W \cap W' \neq \emptyset$. Then $S = \mathbb{R}^2$ and $W = P_c$, $W' = P_{c'}$, where c and c' are suitable geodesics in S. Let π_W be the orthogonal projection onto W (comp. [BGS], § 1). We claim that $\pi_W(W') = S$. Therefore let $x \in W'$, then there exists a geodesic c'_x parallel to c' through x. Thus c'_x has bounded distance to W (since c' is contained in $S \subset W$). It follows that $\pi_W(c'_x)$ is a geodesic parallel to c'_x hence parallel to c'. But the geodesics in W which are parallel to c' are all contained in S.

From the fact that $\pi_W(W') = S$ it follows immediately that the normal vectors to W and W' are perpendicular to each other on the set S. The following lemma is a consequence:

Lemma 7. 1. *Through a given point $x \in X$ there are only finitely many singular submanifolds.*

2. *There exists a constant $\varepsilon_0 > 0$ (only depending on the lower curvature bound) with the following property: If W and W' are singular submanifolds and $d^{n-1}(T_x W, T_y W') \leqq \varepsilon_0$ for $x \in W$, $y \in W'$, then either $W = W'$ or $W \cap W' = \emptyset$. Here d^{n-1} is the distance measured in $G_{n-1}(X)$.*

Note that 2 follows from the fact that if $W \cap W' \neq \emptyset$ then near to the intersection S the tangent spaces are definitely separated from each other.

We now consider the set of all singular submanifolds. Let Y be the subset of the Whitney sum $G_{n-3}(X) \oplus G_1(X)$ defined by the following conditions:

Let σ be an $(n-3)$-plane and τ a 1-plane at $x \in X$ then $(\sigma, \tau) \in Y$ if

1. σ is tangent to an $(n-3)$-flat H,
2. P_H is isometric to a product $\mathbb{R}^{n-3} \times Q$ with $\dim Q = 2$,
3. τ is normal to P_H.

In other words $(\sigma, \tau) \in Y$ if $\tau^\perp$ is the tangent space of a singular submanifold whose euclidean factor is tangent to σ.

Since M is assumed to be analytic it is not difficult to check that Y is an analytic set. Next we prove that singular submanifolds are isolated.

Lemma 8. *Let W be a singular submanifold then there does not exist a sequence W_i of singular submanifolds which converge uniformly on compact subsets to W.*

Proof. Let Y be the analytic set defined above. Note that the projection $p\colon Y \to X$ is not necessarily injective, but by Lemma 7.1 every $x \in X$ has only finitely many preimages in Y. This implies that the dimension of Y is $\leqq n$. If there exists a sequence $W_i \to W$, then as in the proof of theorem 3 the dimension of Y is n somewhere. This implies that there exists a one-parameter family W_t, $0 \leqq t \leqq 1$ of singular manifolds. By Lemma 7.2 we can assume (by choosing a subfamily), that $W_t \cap W_{t'} = \emptyset$ for $t \neq t'$. As in the proof of Theorem 3 this family foliates a convex subset Z of X. As in the proof of Lemma 4 one can find a geodesic $g\colon \mathbb{R} \to X$ which is the axis of an isometry $\gamma \in \Gamma$ (i.e. $\gamma g(t) = g(t+\omega)$ for an $\omega > 0$) with $g(0) \in W_0$, $(g(0,\infty)) \subset Z$, $d(g(t), W_0) \to \infty$ and $d(g(t), W_1) \to \infty$. Thus for given $R > 0$ there exists $m \in \mathbb{Z}$ such that the ball $B_R(g(m\omega)) \subset Z$. Therefore $\gamma^{-m} Z$ contains $B_R(g(0))$. Since $\gamma^{-m} Z$ is foliated by singular submanifolds we obtain in the limit $(m \to \infty)$ a foliation of X by singular submanifolds. By Lemma 7.1 there are only finitely many different foliations of X by singular submanifolds. Thus a subgroup Γ^* of finite index in Γ leaves a foliation invariant and X splits a euclidean factor by Lemma 4. □

Now we can prove Theorem 1 under the additional assumption that M is compact. Let W be a singular submanifold and $\Gamma_W = \{\gamma \in \Gamma \mid \gamma W = W\}$. If W/Γ_W is not compact, then the sets γW, $\gamma \in \Gamma$ are not discrete which contradicts Lemma 2. Thus W/Γ_W is compact and by Theorem 3 there exists an immersed $(n-1)$-torus in W/Γ_W and thus also a codimension 2 torus in M.

It remains to consider the finite volume case. We started from a flat $F \subset X$ such that $\pi(F)$ is contained in a compact subset of M and we constructed a singular submanifold W. It follows from the construction that W contains the limit of $\gamma_j F$, where γ_j is a sequence of elements in Γ. Thus also W contains an $(n-2)$-flat whose image lies in a compact subset of M. Thus we can assume without loss of generality that $F \subset W$. Let $W = \mathbb{R}^{n-3} \times Q$ be the decomposition of W, then F splits as $\mathbb{R}^{n-3} \times \mathbb{R}$, where the second factor is a geodesic in Q.

Let $\Gamma_W = \{\gamma \in \Gamma \mid \gamma W = W\}$. If W/Γ_W has finite volume then the theorem follows by the codimension 1 result of Theorem 3. Thus we may assume that W/Γ_W does not have finite volume. By Lemma 1 we see that for given $\varepsilon > 0$ there is a point $x \in W$ such that $T_x W$ is ε-close to a tangent space of a different singular submanifold. Since we are in a noncompact manifold this does not contradict to the discreteness of singular submanifolds (Lemma 8).

Since $\pi(F)$ is contained in a compact subset of M, also $\pi(\mathbb{R}^{n-3} \times \{q\})$ lies in a compact subset for every $q \in Q$. Every element $\gamma \in \Gamma_W$ respects the splitting of W and operates as (γ_1, γ_2) with $\gamma_1 \in \operatorname{Iso}(\mathbb{R}^{n-3})$ and $\gamma_2 \in \operatorname{Iso}(Q)$. Let $p_i(\gamma) := \gamma_i$ be the projections onto the factors. We consider first the case:

$p_2(\Gamma_W)$ is a *discrete* group of isometries on Q.

We then choose a point $q \in Q$ and $\varepsilon > 0$ with the following properties:

1. If $\gamma = (\gamma_1, \gamma_2) \in \Gamma_W$ with $d(\gamma_2 q, q) \leqq \varepsilon$, then $\gamma_2 = \mathrm{id}$.

2. There does not exist a singular submanifold $W' \neq W$ with $q \in W'$.

This choice is possible since $\pi(\Gamma_W)$ is discrete and since the singular submanifolds are discrete by Lemma 8. From the description of the intersection of singular submanifolds it follows that $D := \mathbb{R}^{n-3} \times \{q\}$ does not intersect any other singular manifold. By choosing $\varepsilon > 0$ small enough we may assume that $d(D, W') \geqq 2\varepsilon$ for all singular $W' \neq W$.

Let H be the ε-tube of D. Let $x \in H$ and $\gamma x \in H$ for a $\gamma \in \Gamma$. By construction $\gamma \in \Gamma_W$ and $\pi_1(\gamma) = \mathrm{id}$. It follows that the kernel of p_2 operates as a Bieberbach group with compact quotient on D. Then there are $(n-3)$ linearly independent translations $\alpha_1, \ldots, \alpha_{n-3}$ in this kernel. We can assume without loss of generality that these elements preserve the two sides of W. Let A be the maximum of the displacement of the elements α_i on W. As discussed above for given $\eta > 0$ there exists $x_0 \in W$ and a singular submanifold $W' \neq W$ such that $d^{n-1}(T_{x_0} W, T_y W') \leqq \eta$ for a suitable $y \in W'$. Since there exists a lower curvature bound $-b^2$, we can assume (by choosing η sufficiently small), that $d(x, W') \leqq \varepsilon(b)$ for all $x \in W$ with distance $\leqq A$ from x_0 where $\varepsilon(b)$ is the constant of Lemma 5. It follows from this lemma that W' can be written canonically as a graph over the ball $B_A(x_0)$ and we can assume that there is no singular manifold "between" W and W' on this ball. By construction the elements α_i leave W' invariant and thus they have an axis also in W' (by [BGS], 6.4). By Lemma 7 we see that $W \cap W' = \emptyset$, hence these axes are not contained in W. Thus the set of axes of α_1 has dimension $> n-1$ and hence X splits a euclidean factor.

Finally we have to consider the case that $p_2(\Gamma_W)$ is not discrete. Let $G = \mathrm{cl}_0(p_2(\Gamma_W))$ be the connected component of the identity of the closure of this group. Since $\dim Q = 2$ we have $\dim G \leqq 3$. If $\dim G \geqq 2$, then one checks easily that G operates transitively on Q. We know already that the image of the euclidean factor of W in M stays in a compact subset. The transitivity of G implies that $\pi(W)$ is contained in a compact subset. It follows that W/Γ_W has finite volume since the singular submanifolds are discrete (Lemma 8). Thus we obtain an $(n-2)$-torus by Theorem 3.

It remains the case $\dim G = 1$, in particular G is abelian. Then $p_2(\Gamma_W)$ has a normal abelian subgroup. From [BGS], § 6, we have the following alternative.

Either all elements of $p_2(\Gamma_W)$ are hyperbolic with a common axis c, or all elements of $p_2(\Gamma_W)$ are parabolic, fix a point $z \in Q(\infty)$ and leave the horospheres at this point invariant.

In the first case Γ_W operates with compact quotient on $F' = \mathbb{R}^{n-3} \times \{c\}$ which then covers a flat torus. We finally show that the case: the elements of $p_2(\Gamma_W)$ are parabolic, cannot occur. Note that there exists $F \subset W$ with $\pi(F)$ stays in a compact subset of M. $F = \mathbb{R}^{n-3} \times \{g\}$, where g is a geodesic in Q. Let b_z be the Busemann function at the fixed point $z \in Q(\infty)$. If b_z is constant on g, then g bounds a flat halfplane and by analyticity Q is flat. Thus b_z is not constant on g. For a suitable orientation of g we have $(b_z \circ g)'(0) = a > 0$. By convexity $b_z(g(t)) \geqq at$ for all $t \geqq 0$. We consider the geodesic $g^*(t) = (p_0, g(t)) \subset W \subset X$ for a fixed point $p_0 \in \mathbb{R}^{n-1}$. Note that $\pi(g^*)$ stays in a compact subset of M.

Consider the points $q_i := g^*(i)$, $i \in \mathbb{N}$ and choose $\gamma_i \in \Gamma$ such that $\gamma_i q_i$ stay in a fixed compact subset of X. Since the singular submanifolds are discrete (by Lemma 8) we can find a subsequence $i \in I$, such that the spaces $\gamma_{i*} T_{q_i} W$, $i \in I$ are tangent to a fixed singular submanifold W_0. It follows that $\gamma_i^{-1}\gamma_j \in \Gamma_W$ for $i, j \in I$. Thus for suitable $\gamma = \gamma_i^{-1}\gamma_j \in \Gamma_W$ we have $\gamma g^*(j)$ is near to $g^*(i)$. This contradicts to the invariance of b_z under $p_2(\Gamma_W)$.

7. Generalisations, examples and final remarks

1. Let $M = X/\Gamma$ be compact and analytic. If X contains a 2-flat which does not cover a torus, then by Proposition 1 there exists a singular submanifold $W = \mathbb{R} \times Q$ in X.

If W does not cover a compact manifold then there exists an accumulation $W_i \longrightarrow W_\infty$ of singular submanifolds. One might hope for a generalization of Proposition 1 to this case. However in the proof of Proposition 1 we have used in an essential way that the converging manifolds F_i are flat and not only the fact that they are totally geodesic.

Consider e.g. the 5-dimensional symmetric space $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$. There are accumulating 2-flats $F_i \longrightarrow F_\infty$ in X (and of course a uniform isometry group). Thus the proposition proves the existence of singular submanifolds $W^3 = \mathbb{R} \times Q^2$. These are the boundary components of X. In X there are also converging boundary components $W_j \longrightarrow W_\infty$. But from this we cannot conclude that there exists a 4-dimensional totally geodesic submanifold in X. (There does not exist a totally geodesic hypersurface in X.) Therefore one cannot hope for an "easy" generalisation of this argument to higher codimension.

2. However one can generalize Proposition 1 in the following way.

Proposition 2. *Let $W_i \subset X$ be complete totally geodesic submanifolds isometric to $\mathbb{R}^k \times Q^s$, $k \geqq 2$, $s \geqq 0$ converging to W. Let the isometry group of X be uniform on W. Then there exists W^* isometric to $\mathbb{R}^k \times Q^s$ and a $(k-1)$-flat $H^* = H' \times \{q\}$ in W^*, where H' is a hyperplane in the euclidean factor, which has a parallel outside of W^*. In particular* $\dim P_{H^*} > \dim W$.

Proof. Consider $p \in W$ and $f_i = d(\ ,W_i)_{|W}$ and copy the proof of Proposition 1 to obtain a point $q_i \in \partial B_i$ with $f_i(q_i) = 1$. Choose a $(k+1)$-flat F_i which contains p and q_i (for the proof we can assume $s \geqq 2$). Then $F_i = \mathbb{R}^k \times \{c_i\}$, where c_i is a geodesic in Q. Now $B_i \cap F_i$ is the ball of radius A_i in F_i. Thus consider in $q_i \in F_i$ the hyperplane $H_i \subset F_i$ tangent to $\partial B_i \cap F_i$. Pulling back everything to a compact fundamental domain we obtain in the limit a manifold W^* isometric to $\mathbb{R}^k \times Q^s$ with a $(k+1)$-flat $F^* = \mathbb{R}^k \times \{c\}$ and a hyperplane (i.e. a k-flat) $D^* \subset F^*$ which has an additional parallel outside of W^*. In particular the $(k-1)$-flat $H^* = D^* \cap \mathbb{R}^k \times \{c^*(0)\}$ has parallels outside of W^*. □

This enables us to prove Corollary 2:

Proof of Corollary 2. We prove that there exists some complete totally geodesic submanifold $W \subset X$ with W isometric to $\mathbb{R}^s \times Q$, $s \geqq 1$ such that W/Γ_W is compact. Then a finite index subgroup Γ_W^* of Γ_W splits as $\mathbb{Z}^s \times \Delta_Q^*$ (compare section 2) and this group contains an abelian subgroup of higher rank. Then M contains a flat torus by the flat torus theorem.

To prove the existence of W we start with a k-flat F in X, $k \geqq n/2$. If F does not cover a flat torus then by Proposition 1 there exists a singular submanifold W_1 isometric to $\mathbb{R}^{k-1} \times Q_1$, $\dim Q_1 \geqq 2$. If W_1/Γ_{W_1} is not compact then by Proposition 2 we obtain a singular submanifold W_2 isometric to $\mathbb{R}^{k-2} \times Q_2$, $\dim Q_2 \geqq 4$. Thus by induction we either obtain some singular submanifold $W = \mathbb{R}^s \times Q$ with $s \geqq 1$, $\dim Q \geqq 2$ with W/Γ_W compact or we obtain a singular submanifold $W_{k-1} = \mathbb{R} \times Q_{k-1}$, with $\dim Q_{k-1} \geqq 2(k-1)$ such that $W_{k-1}/\Gamma_{W_{k-1}}$ is not compact. Since $W_{k-1} \neq X$ and $k \geqq n/2$ it follows that W_{k-1} is a hypersurface.

We can modify the arguments in the proof of Theorem 1 to show that the whole manifold splits an euclidean factor. Then we can use $W = X$. Therefore one has to prove a generalisation of Lemma 7.

Note that $W_{k-1} = P_c$ for a suitable geodesic c (if $\dim P_c > \dim W_{k-1}$, then X splits a euclidean factor). Let $P_c = \mathbb{R} \times Q$ with $\dim Q = n-2$. We call such a hypersurface singular. If Q has an euclidean factor, then P_c has a 2-dimensional $\mathbb{R}$-factor and X splits also an euclidean factor by Proposition 2. Thus we may assume that Q has no euclidean de Rham factor. As in section 6 we study the possible intersection $S = P_c \cap P_{c'}$ of different singular submanifolds. Since Q has no euclidean factor, it does not allow a global parallel vectorfield and the argument in section 6 shows that either

$$S = \{p\} \times Q \text{ or } S = \mathbb{R} \times H,$$

where p is some point in $\mathbb{R}$ or H is a hypersurface in Q. Note that S has a euclidean factor only in the second case. Thus the splitting of S is of the same type in $P_{c'}$.

Thus if $S = \{p\} \times Q$, then P_c and $P_{c'}$ are foliated by parallels to Q and X splits as $X = X' \times Q$ with a twodimensional factor X'. It is standard to reduce this product situation.

If $S = \mathbb{R} \times H$, then S contains a parallel to c and since the splitting is of the same type in $P_{c'}$, also a parallel to c'. It follows with exactly the same arguments as in section 6 that P_c and $P_{c'}$ have orthogonal normal vectors at the intersection.

Thus we can prove an analogue of Lemma 7 and of Lemma 8 which now says that the existence of a converging sequence $\gamma_i W_{k-1}$ implies an euclidean factor of X.

$\square$

3. It is difficult to construct compact analytic rank-1 manifolds (see [BE]) which contain flats of higher dimension. The only example we know is based on a C^∞-construction of Heintze (compare also section 3). Take two noncompact hyperbolic manifolds of finite volume with one cusp and glue them together on the cusp. In the joining cylinder $(-1, 1) \times \mathbb{R}^{n-1}/\Gamma$ choose a warped product metric with a warping function $g(t) = \cosh(t)$. One can extend this metric to a C^∞-metric with $K \leqq 0$ such that $K < 0$ outside of the compact flat totally geodesic submanifold $\{0\} \times \mathbb{R}^{n-1}/\Gamma$. The metric is analytic in a neighborhood of this submanifold. Using methods from sheaf theory (compare [BG] for details) one can approximate the C^∞-metric g_∞ in the C^∞-topology by an analytic metric g_ω, such that g_ω coincides up to order k on the submanifold $\{0\} \times \mathbb{R}^{n-1}/\Gamma$. This is enough to control the curvature everywhere and to show that g_ω has nonpositive curvature.

In all known examples of rank-1 manifolds which are analytic and of finite volume (or even compact C^∞-manifolds) the maximal dimension of a flat in X is either 1 or $n-1$. However we should remark that the methods of this paper work in a larger category of spaces:

Consider analytic manifolds H/Γ with $K \leqq 0$ and possibly with boundary such that the universal covering H of M is convex and the curvature is strictly negative in a neighborhood of the boundary ∂H. Note that in the proofs above we only need the completeness of submanifolds with local euclidean factors. If the curvature is strictly negative near the boundary then such a submanifold cannot reach the boundary and is therefore complete. In this category there are much more examples. Take any higher rank manifold W of finite volume and a disk D and consider a warped product $D \times_g W$ where g is a strictly convex function on D with a minimum at the origin.

4. Propositions 1 and 2 may be useful in the study of homogeneous manifolds with $K \leqq 0$ which contain flats of higher dimension.

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